PROCEEDINGS

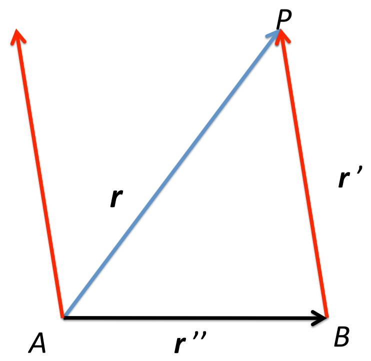
Introduction to Cosmology

ABSTRACT: An introductory note of the talk on cosmology. Some of the text are copied from the talk by G. Lazarides, http://arxiv.org/pdf/hep-ph/9904502v2.pdf.

KEYWORDS: cosmology.

1. Redshift and Hubble Constant

1.1 Redshift due to the recessional velocity



From homogeneity and isotropy of any expanding universe, one can show that the recessional velocity $\mathbf{v_B}$ of the (galaxies) photon at P observed by observer B is the same as the recessional velocity $\mathbf{v_A}$ of the photon at the same direction observed by observer B, namely:

$$\mathbf{v}_B(\mathbf{r}', t) = \mathbf{v}_A(\mathbf{r}', t). \tag{1.1}$$

Assume there is a small relative velocity of B frame, $\mathbf{v}_A(\mathbf{r}'', t) < c$, as seen from the A frame along the radius vector $\mathbf{r}'' = \mathbf{r} - \mathbf{r}'$. Therefore the velocity $\mathbf{v}_A(\mathbf{r}', t)$ can be expressed as:

$$\mathbf{v}_B(\mathbf{r}', t) = \mathbf{v}_A(\mathbf{r}, t) - \mathbf{v}_A(\mathbf{r}'', t) = \mathbf{v}_A(\mathbf{r}', t) \ .$$

Therefore,

$$\mathbf{v}_A(\mathbf{r}', t) = \mathbf{v}_A(\mathbf{r}, t) - \mathbf{v}_A(\mathbf{r}'', t) \ , \tag{1.2}$$

$$\mathbf{r}' = \mathbf{r} - \mathbf{r}'' \tag{1.3}$$

is true for every point P in any isotropic and homogeneous space. This hence implies that:

$$\mathbf{v}_A(\mathbf{r}, \mathbf{t}) = \mathbf{f}(\mathbf{t})\mathbf{r} \ . \tag{1.4}$$

Namely, the velocity-distance ratio is a constant in $\mathbf{r}$.

Redshift z of the photon spectrum from distant universe at distance r due to the recessional velocity v of the photon field is related by:

$$1 + z = \sqrt{\frac{1 + v/c}{1 - v/c}}. \tag{1.5}$$

Hence we can show that

$$v/c \sim z \tag{1.6}$$

for $v \ll c$. For any expanding space with the relation

$$v = H_0 r, \tag{1.7}$$

the redshift can be shown to follow the relation:

$$z \sim H_0 \frac{r}{c} \tag{1.8}$$

1.2 Lorentz Transformation

The Lorentz transformation is given by

$$x'^a = \Lambda^a{}_b x^b \ , \tag{1.9}$$

with

$$(\ \Lambda\ )^a{}_b = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.10}$$

for two reference frames moving apart with velocity v and $\beta = v/c$.

In fact all 4-vectors A^a transform as $x^a = (ct,\ \mathbf{x})$, namely,

$$A'^a = \Lambda^a{}_b A^b \ . \tag{1.11}$$

For example, the 4-momentum vector $k^a = (\omega/c,\ \mathbf{k})$ is also a 4-vector derived from the identity $k_a = i\partial_a = i(\partial_t/c, \nabla_i) = (\omega/c,\ -\mathbf{k})$. This also follows from the fact that the plane wave phase factor takes the form $\exp[-ik_a x^a]$. Therefore

$$k'^a = \Lambda^a{}_b k^b \ . \tag{1.12}$$

Hence $k' = \gamma(-\beta\omega/c + k)$ with k the momentum 3-vector in x-direction. This leads to the result

$$k' = k\gamma(1-\beta) = k\sqrt{\frac{c-v}{c+v}} \ , \tag{1.13}$$

following from the fact that $c = \omega/k$. Note that λ is the wavelength of the photon in the source rest frame. λ is the photon wavelength in observer's rest frame. Note that

$$1 + z \equiv \frac{\lambda_{\rm obs}}{\lambda_{\rm source}} = \frac{\lambda'}{\lambda} = \frac{k}{k'} \ . \tag{1.14}$$

Hence the identity of red-shift formula follows:

$$1 + z = \sqrt{\frac{1+v/c}{1-v/c}} \ . \tag{1.5}$$

1.3 Redshift due to the Recessional Velocity relative to $R(t)$

For an expanding universe with a scale factor $R(t)$, we can define a dimensionless parameter $a(t)$:

$$a(t) = \frac{R(t)}{R_0}. \tag{1.15}$$

We will show in a moment that the peculiar velocity and its associated momentum of any matter field in the FRW space is proportional to $1/R(t)$. $p \sim 1/R$ is also true for photon field. Hence the wavelength of the photon source $\lambda = h/p \sim R$ obeys the relation:

$$\frac{\lambda_0}{R_0} = \frac{\lambda}{R(t)}. \tag{1.16}$$

In addition, a can be expanded as:

$$a(t) = 1 - \dot{a}_0(t_0 - t) \tag{1.17}$$

by assuming the expansion rate is small. Exact relation will be shown later when we start to solve the gravitational field equations in the FRW space with various matter fields incorporated in a consistent manner.

From the definition of z

$$z = \frac{\lambda_0 - \lambda}{\lambda} = \frac{1}{a} - 1 \ ,$$

one can show that

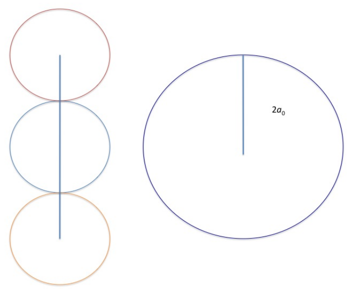
$$z \sim \dot{a}_0 \frac{r}{c} \ , \tag{1.18}$$

with the expansion coefficient $\dot{a}_0$ related to the Hubble parameter H_0:

$$\dot{a}_0 = \frac{\dot{R}_0}{R_0} = H_0 \ . \tag{1.19}$$

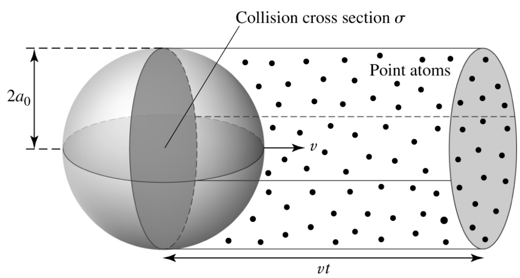
This shows that the definition of redshift (1.5) agrees with the definition $1+z = 1/a(t) = R_0/R(t)$.

2. Cross section



Cross section is the area of the effective collisional impact. For a classical atom with Bohr radius a_0, the cross section

$$\begin{aligned} \sigma &= \pi(2a_0)^2 \ , \qquad (2.1) \\ a_0 &= \frac{\hbar}{\alpha\ m_e c} \ , \\ \alpha &= \frac{e^2}{(4\pi\epsilon_0)\hbar c} \sim \frac{1}{137} \ . \end{aligned}$$



The mean free path l of a interacting particle is the distance between two effective collisions. Assuming a particle moving with velocity v can travel a distance L in time interval t such that $L = vt$, $\sigma l = V_{\text{one-collision}}$. $lN = L$ follows if there are N particles in the volume $V = \sigma L$. For a particle with number density $n = N/V$, one can thus show that

$$l = \frac{1}{n\sigma}. \tag{2.2}$$

This follows from the fact that:

$$n = \frac{N}{V} = \frac{L/l}{\sigma L} = \frac{1}{\sigma l} \; .$$

Note that $[n\sigma] = [1/L^3][L^2] = [1/L]$ gives the correct dimension for $1/l$.

2.1 Interaction Rate

The interaction rate Γ is defined as the number of interactions in unit time:

$$\Gamma = n\sigma v \tag{2.3}$$

with v the speed of the particle. Another way to look at this definition is:

$$\Gamma l = v \; . \tag{2.4}$$

Mean free path times the interaction per unit time is the velocity of the particle. In addition, the dimension of the interaction rate is:

$$[\Gamma] = [\frac{1}{L^3}][L^2][\frac{L}{t}] = [\frac{1}{t}] \tag{2.5}$$

which is consistent with its definition.

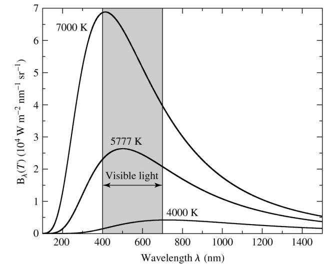
For example, the photon interaction rate

$$\Gamma_\gamma = n_e \sigma_T c \tag{2.6}$$

with $\sigma_T = 6.65 \times 10^{-25}\text{cm}^2$ the Thompson cross section. The cross section of different interactions have to rely on the fundamental quantum field theories known to most particle physicists. We will not go into the details of the detailed derivations. The results will be given. These results can also be checked out from the well known data book or any particle physics textbook.

In particular, when $\Gamma > H$, particle interaction is still vivid as the interaction rate is greater than the expansion rate. When $\Gamma < H$, particle interaction will gradually decouple from the heat bath as we will return to this topic shortly.

3. blackbody radiation



Blackbody radiation assures that all photons are in thermal equilibrium via collisions with charged medium and obeys the Planck distribution function of the form:

$$B_\lambda(T) = \frac{2hc^2/\lambda^5}{\exp[hc/\lambda kT] - 1} \tag{3.1}$$

Express $B_\lambda d\lambda = -B_\nu d\nu$ as a distribution function of the frequency ν, it is apparent that:

$$B_\nu(T) = \frac{2h\nu^3/c^2}{\exp[h\nu/kT] - 1}. \tag{3.2}$$

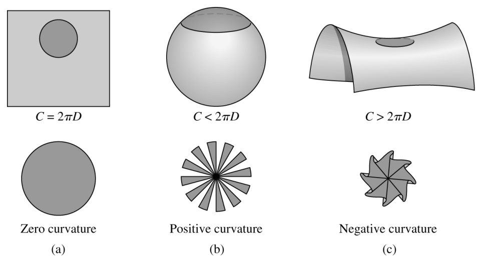
Counting the dimension:

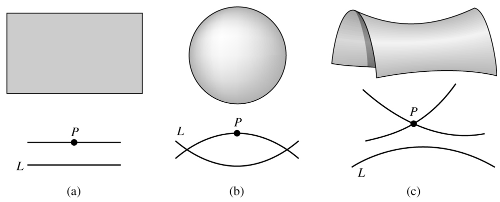
$$[B_\lambda] = [hc^2/\lambda^5] = \left[\frac{E}{tL^3}\right] \tag{3.3}$$

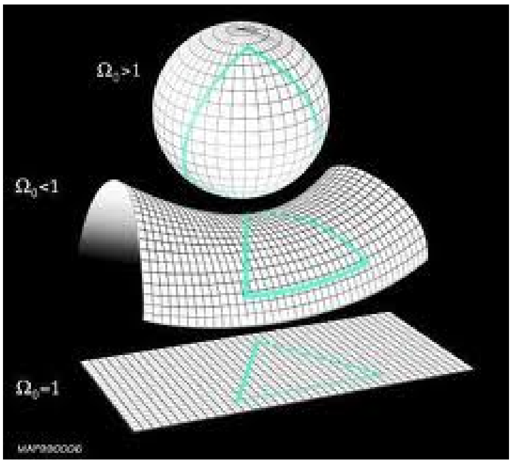
is the dimension of the power density. Note that the energy density is $u_\lambda d\lambda = (4\pi/c)B_\lambda d\lambda$, and

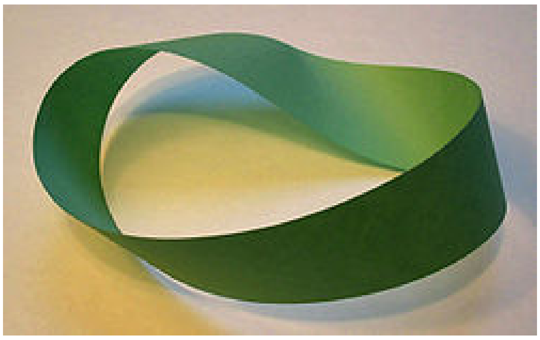
$$[u_\lambda d\lambda] = [hc/\lambda^4] = \left[\frac{E}{L^3}\right] \; . \tag{3.4}$$

4. Friedmann-Robertson-Walker metric space









Homogeneous and isotropic 3-spaces can be classified into three different classes that can be parametrized as:

$$X^2 + Y^2 + Z^2 + W^2 = A^2 \qquad (4.1)$$
$$X^2 + Y^2 + Z^2 = B^2 \qquad (4.2)$$
$$X^2 + Y^2 + Z^2 - W^2 = A^2 \qquad (4.3)$$

for a constant scale factor A parameterizing the radius of the 3-space. B can be arbitrary radius for the space $\mathbf{R}^3$. The first class is a closed space, the second class is a flat space and the last one is the open space. It can be parametrized alternatively as:

$$ds_3^2 = g_{ij}dx^i dx^j = a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\ d\varphi^2)\right] , \qquad (4.4)$$

with r, φ and θ 'comoving' polar coordinates. The parameter k is the 'scalar curvature' of the 3-space. $k = 0$, $k > 0$ or $k < 0$ correspond to flat, closed or open universe. The dimensionless parameter $a(t)$ is the 'scale factor' of the universe normalized by taking $a(t)$ in unit of R_0. Equivalently, we are taking $R(t) = a(t)R_0 \equiv a(t)R(t_0)$. Here $R(t)$ is the scale factor of the universe and t_0 is the present cosmic time. Hence $a_0 = 1$, i.e. one unit of R_0.

Eq. (4.4) can be proved by parameterizing, for example, the closed 3-space as;

$$X = A\sin\chi\sin\theta\sin\varphi, \qquad (4.5)$$
$$Y = A\sin\chi\sin\theta\cos\varphi, \qquad (4.6)$$
$$Z = A\sin\chi\cos\theta, \qquad (4.7)$$
$$W = A\cos\chi. \qquad (4.8)$$

It then follows that

$$dX^2 + dY^2 + dZ^2 + dW^2 = A^2\left[d\chi^2 + \sin^2\chi d\Omega\right]. \qquad (4.9)$$

This gives the case $k = 1$ with the parameterization $r = \sin\chi$. Indeed, $dr = \cos\chi d\chi$, and hence $dr^2/\cos^2\chi = d\chi^2 = dr^2/(1-r^2)$. Similarly for the open space with $k = -1$ by replacing $\sin\chi$ ($\cos\chi$) with $\sinh\chi$ ($\cosh\chi$).

The four dimensional spacetime in the universe is then described by the Friedmann-Robertson-Walker metric

$$ds^2 = dt^2 - ds_3^2. \qquad (4.10)$$

5. geodesic equation of a test particle

For a particle traveling on a metric space given by:

$$ds^2 = g_{ab}dx^a dx^b , \qquad (5.1)$$

the trajectory will make the length of the trajectory as short as possible. Therefore, the trajectory can be derived from the least action principle with action given by the length $\int ds$. It is also equivalent to varying the Lagrangian given by

$$\mathcal{L} = \sqrt{g_{ab}\frac{dx^a}{dt}\frac{dx^b}{dt}} . \qquad (5.2)$$

The derivation is straightforward:

$$\delta\mathcal{L} = \frac{1}{2\mathcal{L}}\left(\partial_c g_{ab}\delta x^c v^a v^b + 2g_{ac}v^a\frac{d}{dt}\delta x^c\right) = \frac{1}{2}\left[\partial_c g_{ab}u^a v^b - 2\frac{d}{dt}(g_{ac}u^a)\right]\delta x^c \qquad (5.3)$$

after an integration-by-part. Note that $u^a = (u^0, u^i)(= \gamma(1, v^i)$ for the Minkowski space with metric $g_{ab} = \eta_{ab}$) with $u^a = dx^a/ds$ and $v^a = dx^a/dt$. Therefore the field equation leads to:

$$\dot{u}^a + \frac{1}{2} g^{ad}(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}) u^b v^c$$
$$\equiv \dot{u}^a + \Gamma^a_{bc} u^b v^c = 0 \ . \tag{5.4}$$

Note that g^{ab} is the inverse of the metric field satisfying $g^{ab} g_{bc} = \delta^a_c$. It is also true if we take another affine parameter λ to replace t:

$$\frac{du^a}{d\lambda} + \Gamma^a_{bc} u^b \frac{dx^c}{d\lambda} = 0. \tag{5.5}$$

Taking λ as s:

$$\frac{du^a}{ds} + \Gamma^a_{bc} u^b u^c = 0 \tag{5.6}$$

This equation is referred to as the geodesic equation.

5.1 covariant derivative

Note that the geodesic equation can also be written as

$$du^a + \Gamma^a_{bc} u^b dx^c$$
$$= (\partial_c u^a + \Gamma^a_{bc} u^b) dx^c = 0 \ . \tag{5.7}$$

Hence the geodesic equation becomes

$$D_c u^a = \partial_c u^a + \Gamma^a_{bc} u^b = 0 \tag{5.8}$$

with the operator $D_c u^a$ known as the covariant derivative of any contra-variant vector u^a.

Note that the covariant derivative of any covariant vector v_a is

$$D_c v_a = \partial_c v_a - \Gamma^b_{ca} v_b \ . \tag{5.9}$$

This can also be derived by requiring that

$$D_c(u^a v_a) = \partial_c(u^a v_a) \ , \tag{5.10}$$
$$D_c(u^a v_a) = (D_c u^a) v_a + u^a (D_c v_a) \ . \tag{5.11}$$

Note that the first requirement is demanding that $u^a v_a$ is a 4-scalar. The second requirement is the Leibniz rule.

Derivative operators ∂_c is known as the translational operators. Indeed, we can write

$$\exp\left[i\frac{P}{\hbar}(x+a)\right] = [1 + a\partial_x] \exp\left[i\frac{P}{\hbar}x\right] \ . \tag{5.12}$$

It is equivalent to parallel transport a physical observable like a vector v_b form one point to another nearby point. If the space is curved, parallel transport is defined in conjunction to the curved geometry. Normal derivative will carry the vector off its original space and no longer remains a well-defined vector in its original space. Covariant derivative D_c is therefore designed to remove off-space component of the transported vector, namely, $\Gamma^b_{ca} v_b$ from the normal derivative. After removing unphysical component of the transported vector, we will be able to defined a new transported vector defined on our own space time.

By requiring

$$D_c g_{ab} = \partial_c g_{ab} - \Gamma^d_{ca} g_{db} - \Gamma^d_{cb} g_{ad} \tag{5.13}$$

as if it is similar to the covariant derivative of a product of two covariant vectors $A_a B_b$. Note that

$$D_c(A_a B_b) = (D_c A_a) B_b + A_a (D_c B_b) \tag{5.14}$$

obeying the Leibniz rule. Then it is straightforward to show that

$$D_c g_{ab} = 0 \tag{5.15}$$

which is known as the compatibility condition in Riemannian geometry. We will come back to the details of the tensor calculus later in this text.

6. particle horizon and velocity fields

$ds^2 = 0$ for a photon field on the FRW space. Hence

$$\int_0^t \frac{dt'}{a(t')} = \int_0^{r_H} \frac{dr}{\sqrt{1 - kr^2}}. \tag{6.1}$$

Hence the particle horizon

$$d_H = \int_0^{r_H} dr \sqrt{g_{rr}} \tag{6.2}$$

can be shown to be:

$$d_H = a(t) \int_0^{r_H} \frac{dr}{\sqrt{1 - kr^2}}$$
$$= a(t) \int_0^t \frac{dt'}{a(t')}. \tag{6.3}$$

Writing in conformal coordinate:

$$ds^2 = g_{ab}dx^a dx^b$$
$$= a^2(\eta)\left[d\eta^2 - \frac{dr^2}{1-kr^2} - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right], \tag{6.4}$$

we can write the proper distance as

$$d_H = a(t)(\eta(t) - \eta_0). \tag{6.5}$$

6.1 Peculiar Velocity

Peculiar velocity is defined as the velocity with respect to the co-moving frame. Therefore the peculiar 4-velocity $u^a = dx^a/ds$ obeys the geodesic equation:

$$\frac{du^a}{d\lambda} + \Gamma^a_{bc}u^b\frac{dx^c}{d\lambda} = 0 \tag{6.6}$$

with λ some affine parameter. Note that $u^a = (u^0, u^i) = \gamma(1, v^i)$ with $v^i = dx^i/dt$ the 3-velocity.

The 0-component geodesic equation reads:

$$\frac{du^0}{ds} + \Gamma^0_{bc}u^b u^c = 0, \tag{6.7}$$
$$\frac{du^0}{ds} + H\mathbf{u}^2 = 0. \tag{6.8}$$

Eq. (6.8) can also be derived directly from the pre-arranged geodesic equation (5.3)

$$\partial_t g_{ab}u^a v^b - 2\frac{d}{dt}(g_{at}u^a) = 0\ . \tag{6.9}$$

Indeed, it reproduces: $2(Hu^2 + du^0/ds) = 0$. From the fact that ${u^0}^2 - |\mathbf{u}|^2 = 1$, hence $u^0du^0 = \mathbf{u}\cdot d\mathbf{u}$, for any matter field, one can derive

$$\frac{du}{u^0ds} + Hu = 0 \tag{6.10}$$

with $u \equiv |\mathbf{u}|$. Note that $u^0ds = dt$ following the definition of u^0. Therefore, the geodesics equation gives

$$\frac{\dot{u}}{u} = -H = -\frac{\dot{a}}{a}. \tag{6.11}$$

This implies immediately that

$$u = \frac{p}{m} \propto \frac{1}{a} \to 0 \tag{6.12}$$

at time infinitive for any expanding universe. Note that $p \propto 1/R$ is also true for photon field. Hence we reach the conclusion that the photon wave length is proportional to the scale factor $R(t)$ or $a(t)$.

$$\lambda \sim \frac{h}{mu} \sim a(t). \tag{6.13}$$

Hence the redshift can be defined also as $1+z = a_0/a$ as mentioned earlier.

6.2 luminosity distance

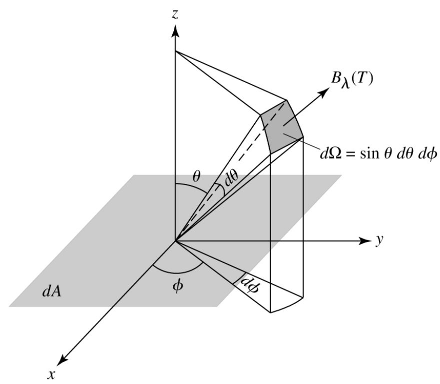
The flux distribution F_λ has a dimension given by $[F] = [L/A] = [E/tL^2]$. It is a power per unit area. Equivalently, the flux follows

$$F_\lambda d\lambda = \frac{L_\lambda d\lambda}{4\pi r^2} = B_\lambda d\lambda\frac{R^2}{r^2} \tag{6.14}$$

for a radiation source with B_λ radiating from the source sphere with radius R. r given above is the distance from the radiation source to the detector. For an expanding universe, the radiation source far away is red-shifted by a factor $1+z$. There is another redshift factor $a+z$ representing the power received by the detector due to the time-lag effect derived from $[F] = [dE]/[A][dt]$. Therefore, the flux received by the detector will be:

$$F = \frac{L}{4\pi d_L^2} = \frac{L}{4\pi r^2(1+z)^2}. \tag{6.15}$$

$d_L = r(1+z)$ is known as the luminosity distance.



For a power $B_\lambda d\lambda$ emitted with an angle θ to the detector, the total luminosity $L_\lambda d\lambda$ will be:

$$[L_\lambda d\lambda] = [B_\lambda d\lambda][dA\cos\theta][d\Omega]. \tag{6.16}$$

Therefore,

$$L_\lambda d\lambda = \int_0^{2\pi}\int_0^{\pi/2}\int_A [B_\lambda d\lambda][dA\cos\theta][d\Omega]$$
$$= 4\pi R^2 B_\lambda d\lambda \tag{6.17}$$

with R the radius of the radiation source sphere. In addition,

$$\int_0^\infty B_\lambda d\lambda = \frac{\sigma}{\pi} T^4 \ , \tag{6.18}$$

with

$$\sigma = \frac{2\pi^5 k^4}{15 c^3 h^3} \ . \tag{6.19}$$

7. Hubble's law

The proper distance is given by

$$d_H = a(t) \int_0^r \frac{dr}{(1-kr^2)^{1/2}} \ . \tag{7.1}$$

We will write $\mathbf{d}_H = a(t)\mathbf{r}$, for a flat universe ($k=0$), with $\mathbf{r}$ a 'comoving' and $\mathbf{d}_H$ a physical vector in 3-space. Hence the velocity of an object is

$$\mathbf{V} = \dot{\mathbf{d}}_H = \frac{\dot a}{a} \mathbf{d}_H + a \frac{d\mathbf{r}}{dt} \ , \tag{7.2}$$

with over-dots denoting derivation with respect to cosmic time. The second term in the right hand side (RHS) of this equation is the 'peculiar velocity', $\mathbf{v} = a(t)\mathbf{r}$, of the object. It is the velocity with respect to the 'comoving' coordinate system. For $\mathbf{v} = 0$, Eq.(7.2) reads

$$\mathbf{V} = \frac{\dot a}{a} \mathbf{d}_H \equiv H(t) \mathbf{d}_H \ , \tag{7.3}$$

with $H(t) \equiv \dot a(t)/a(t)$ the Hubble constant. This is the well-known Hubble law claiming that everything runs away from each other with velocity proportional to their distances.

8. Conservation Law of the perfect fluid

Homogeneity and isotropy of the universe imply that the energy momentum tensor takes the diagonal form

$$(T^a_{\ b}) = \mathrm{diag}(\rho, -p, -p, -p) \ , \tag{8.1}$$

with ρ the energy density of the universe and p the pressure. Energy momentum conservation

$$D_a T^a_{\ b} = 0 \tag{8.2}$$

can be expressed as the continuity equation

$$\frac{d\rho}{dt} = -3H(t)(\rho + p) \ , \tag{8.3}$$

where the first term in the rhs describes the dilution of the energy due to the expansion of the universe and the second term corresponds to the work done by pressure. Eq.(8.3) can be given the following more transparent form

$$d\left(\frac{4\pi}{3} a^3 \rho\right) = -p \ 4\pi a^2 da \ , \tag{8.4}$$

which indicates that the energy loss of a 'comoving' sphere of radius $\propto a(t)$ equals the work done by pressure on its boundary as it expands.

Note that Eq. (8.4) can be interpreted as a thermal dynamical equation:

$$dU = -pdV + \frac{dS}{T} \ , \tag{8.5}$$

with $U = \rho V$, $V = 4\pi a^3/3$, S the entropy and T the temperature of the thermal dynamical system. $dS = 0$ for a closed system without energy loss to its environment. Note that $dS/T = \bar{d}Q$.

Also note that the volume of the 3-sphere S^3 is $2\pi^2 a^3$ instead of $4\pi a^3/3$. This follows from the metric element shown in Eq. (4.9):

$$ds^2 = a^2 \left[d\chi^2 + \sin^2\chi d\Omega\right] . \tag{8.6}$$

The volume of the 3-sphere is

$$\int_0^\pi d\chi \int_0^\pi d\theta \int_0^{2\pi} d\varphi \ \sqrt{g} \tag{8.7}$$

with $\sqrt{g} = a^3 \sin^2\chi \sin\theta$. The integral can be shown directly to give $2\pi^2 a^3$. Hence the Newtonian approach can not taken seriously at this point.

9. Friedmann Equation

For a universe described by the Robertson-Walker metric in Eq.(4.10), Einstein's equations

$$R_a^{\ b} - \frac{1}{2}\ \delta_a^{\ b} R = 8\pi G \ T_a^{\ b} \ , \tag{9.1}$$

where $R_a^{\ b}$ and R are the Ricci tensor and scalar curvature tensor and $G \equiv M_P^{-2}$ is the Newton's constant, lead to the Friedmann equation

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho \tag{9.2}$$

with $H \equiv \dot a(t)/a(t)$ the Hubble parameter.

Writing the equation in the form:

$$\frac{1}{2}m\dot{a}^2 - mG\left(\frac{4\pi}{3}a^3\right)/a\rho = -m\frac{k}{2}\ , \quad (9.3)$$

one can read this equation as the Newtonian energy conservation law for a test particle m moving under the gravitational attraction due to a sphere of uniform mass density ρ and radius a:

$$T - V = -m\frac{k}{2}\ . \quad (9.4)$$

The velocity $\dot{a}$ is the escape velocity for the case $k = 0$. On the other hand, the system is a bound state for $k > 0$. The test particle will escape when $k > 0$.